

Optimized Distribution of Entanglement Graph States in Quantum Networks

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ABSTRACT Building large-scale quantum computers, essential to demonstrating quantum advantage, is a key challenge. Quantum Networks (QNs) can help address this challenge by enabling the construction of large, robust, and more capable quantum computing platforms by connecting smaller quantum computers. Moreover, unlike classical systems, QNs can enable fully secured long-distance communication. Thus, quantum networks lie at the heart of the success of future quantum information technologies. In quantum networks, multipartite entangled states distributed over the network help implement and support many quantum network applications for communications, sensing, and computing. Our work focuses on developing optimal techniques to generate and distribute multipartite entanglement states efficiently.

Prior works on generating general multipartite entanglement states have focused on the objective of minimizing the number of maximally entangled pairs (EPs) while ignoring the heterogeneity of the network nodes and links as well as the stochastic nature of underlying processes. In this work, we develop a hypergraph-based linear programming framework that delivers optimal (under certain assumptions) generation schemes for general multipartite entanglement represented by graph states, under the network resources, decoherence, and fidelity constraints, while considering the stochasticity of the underlying processes. We illustrate our technique by developing generation schemes for the special cases of path and tree graph states, and discuss optimized generation schemes for more general classes of graph states. Using extensive simulations over a quantum network simulator (NetSquid), we demonstrate the effectiveness of our developed techniques and show that they outperform prior known schemes by up to orders of magnitude.

INDEX TERMS Quantum Communications, Quantum Networks

I. INTRODUCTION

Quantum networks (QNs) enable the construction of large-scale and robust quantum computing platforms by connecting smaller QCs [1]. QNs also enable various important applications [2]–[11], but to implement and support many of these applications, we need to create and distribute entangled states efficiently [12]–[17]. Recent works have addressed the generation of entanglement states but in limited settings, e.g., bipartite and GHZ states, or graph states with a simplistic optimization objective. In this paper, we consider the generation and distribution of specialized graph states over quantum networks, with minimal generation latency, taking into consideration the stochastic nature of the underlying generation process.

Graph States and Their Applications. Graph states are multipartite entangled states where a graph over the qubits

specifies the entanglement structure between qubits. Owing to their highly entangled nature, graph states find applications in various quantum information processing domains, such as measurement-based quantum computing, quantum error correction, quantum secret sharing, and quantum metrology. In particular, path/cycle graph states are used as a primary resource state of fusion-based quantum computing [18], and tree graph states find usage in counterfactual error correction [19], photonic measurement-based quantum computing, and fusion-based quantum computing [18]. The star graph state, which is a special case of a tree graph state, is equivalent to a GHZ state—which has many applications, including error correction [19], quantum secret sharing [20], quantum metrology [21], clock synchronization [5], etc. Therefore, developing efficient generation schemes to distribute graph states in a QN is of great significance. Our work focuses on

developing optimal generation schemes for general classes of graph states.

Prior Work and Our Approach. There have been recent works [22]–[24] that have addressed the problem of efficient generation and distribution of general graph state entanglements in a quantum network. These works, however, have focused on the simplistic optimization objective of minimizing the *number* of maximally entangled pairs (e-bits or EPs) consumed; in particular, they implicitly ignore the stochastic nature of the underlying processes. Even a true *count* of EPs consumed should consider the stochastic nature of operations (e.g., fusion) involved, particularly since they can have a relatively low probability of success. Moreover, some EPs may take significantly longer to generate than others due to the heterogeneity of the network. Thus, the number of EPs is too simplistic a performance metric.

In this work, we consider the generation and distribution of classes of graph states to maximize the expected generation rate under given network resource and fidelity constraints while considering the stochastic nature of underlying processes and network heterogeneity. This is in the same vein as the recent works on the generation of EPs [12], [25]–[27] and GHZ states [28] in quantum networks. In particular, our goal is to develop *provably optimal* generation schemes. We develop a framework—based on a hypergraph representation of the intermediate graph states and fusion operations—that delivers optimal (under reasonable assumptions) generation schemes under network and fidelity constraints. We illustrate our framework by developing multiple generation schemes for the path and tree graph states, and discuss generalizations to other classes of graphs. In essence, our proposed schemes use fusion operations to build larger graph states from smaller ones progressively and discover the optimal level-based structure (that represents the generation process, i.e., sequence and order of fusion operations over intermediate graph states) by using an appropriate linear programming formulation.

Our Contributions. In the above context, we make the following contributions.

- 1) We develop a framework for developing optimal schemes for generating graph states in quantum networks under network resource and fidelity constraints, considering the stochastic nature of the fusion operations.
- 2) Specifically, for path graph states, we design a polynomial-time generation scheme that is provably optimal under reasonable assumptions. In addition, we also develop an optimal two-stage generation scheme that is computationally more efficient, based on restricting the intermediate graph states created.
- 3) Similarly, for tree graph states, we design two generation schemes that are optimal under the restriction on the intermediate states and fusion operations used.
- 4) We show the versatility of our developed scheme by discussing and illustrating its application for other classes

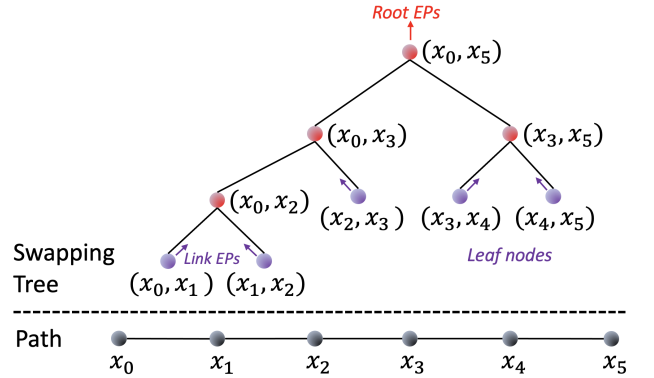


FIGURE 1. A swapping tree over a path. The leaves of the tree are the link EPs, which are being generated continuously. Here, the notation (x_i, x_j) represents an EP over two qubits residing in the network nodes x_i and x_j .

of graph states, e.g., grid graphs, bipartite graphs, and complete graphs. We also generalize our scheme to generate multiple graph states concurrently.

- 5) Using extensive evaluations over the NetSquid simulator, we demonstrate the effectiveness of our developed techniques and show that they outperform prior work by up to orders of magnitude.

II. BACKGROUND

Quantum Network (QN), Nodes, Links, and Communication. A quantum network (QN) is a network of quantum computers (QCs), and is represented as a connected undirected graph with vertices as QCs and edges representing the (quantum and classical) direct communication links. We use *network nodes* to refer to the vertices (QCs) and *links* to refer to the edges, in the QN graph. We discuss a detailed network model in §III. Since direct transmission of quantum data is subject to unrecoverable errors, especially over long distances, we use teleportation to transfer quantum information reliably across nodes in a QN. Teleportation requires that a maximally-entangled pair (EP) be already established over communicated nodes.

Generation of Remote EPs using Swapping Trees. An efficient way to generate an EP over a pair of remote network nodes (s, d) using EPs over network links (i.e., edges) is to: (i) create a path P in the network graph from s to d with EPs over each of the paths' edges, and (ii) perform a series of entanglement swaps (ES) over these EPs. The series of ES operations over P can be performed in any arbitrary order, but this order of ES operations affects the latency incurred in generating the EP over (s, d) . One way to represent the “order” in which the ES operations are executed—is a complete binary tree over the link EPs as leaves, called a *swapping tree* [12]. See Fig. 1 (from [12]). The stochastic nature of ES operations entails that generation of an EP over a remote pair of nodes using a swapping tree may incur significant latency, called the *generation latency* (inverse of generation rate). Generation latency is largely due to the latency incurred in (i) generating the link EPs, and (ii) a generated EP (x_i, x_j) waiting for its “sibling” EP (x_j, x_k) to

be generated before an ES operation can be performed over them to generate an EP over (x_i, x_k) .

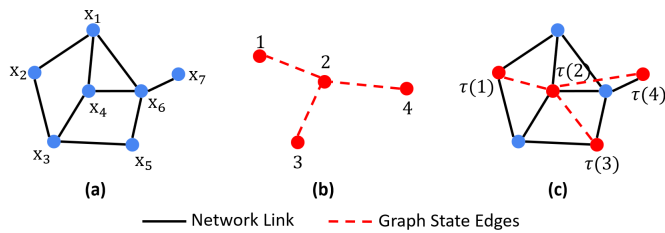


FIGURE 2. (a) A quantum network with 7 nodes $\{x_1, x_2, \dots, x_7\}$, (b) Graph State G with 4 vertices named 1 to 4, and (c) Distributed Graph State (in red) for the graph state G with $\tau(1) = x_2, \tau(2) = x_4, \tau(3) = x_5, \tau(4) = x_7$.

A. GRAPH STATES: MULTIPARTITE ENTANGLEMENTS

Distributed Graph States; Link States. A graph state is a multipartite quantum state $|G\rangle$ which is described by a graph G , where the vertices of G correspond to the qubits of $|G\rangle$. Formally, a graph state $|G\rangle$ is given as

$$|G\rangle = \prod_{(u,v) \in E(G)} C_Z^{(u,v)} \otimes_{v \in V(G)} |+\rangle_v,$$

where $C_Z^{(u,v)}$ is the controlled-Z (CZ) gate over the qubits u and v . We use the term *distributed* graph state to mean a graph state G along with its (target or current) distribution over the given quantum network; this distribution is represented by a function $\tau : V(G) \mapsto V(Q)$ of graph state's vertices $V(G)$ to the nodes $V(Q)$ in the given quantum network Q . See Fig. 2.¹ For brevity, when clear from the context, we just use *states* to refer to *distributed graph states*. Also, we use the term *link states* to refer to the single-edge graph states distributed over the network links; these link states are locally equivalent to the link-EPs generated by the adjacent nodes.

Generation² of Graph States via Fusion Trees. We need to fuse smaller graph states and/or modify graph states to generate general graph states. In general, starting with link states, we want to generate graph states using only *local* quantum operations (i.e., gates with operands in a single node). Similar to swapping trees used to describe the generation of EPs, we can use *fusion trees* to describe the generation of graph states in a QN using local fusion operations. Each node in a fusion tree would represent a distributed graph state. Such fusion trees have been used in prior works—e.g., for generating and distributing GHZ states [28].

Fusion Operations. Local operations within a fusion are generally restricted to single-qubit Clifford operations, local CZ gates, or Pauli measurements. In our context, we only use the following operations or measurements within a local fusion operation (see Fig. 3):

- (a) Create or remove an edge (in the graph state) by doing a CZ operation over two vertices (of the graph state). This

¹We generally use x_i 's and y for network nodes, and numbers 1, 2, etc. for graph state's vertices.

²Throughout the paper, by *generation* of states, we implicitly mean generation and distribution of created states.

- operation is local when the qubits corresponding to the vertices are available in a single node.
- (b) Pauli-Z measurement over a qubit/vertex q results in q 's deletion.
- (c) Pauli-Y measurement over a qubit/vertex q results in a local complementation of vertex q 's neighborhood and then q 's deletion.
- (d) Also, one can effectuate local complementation of any vertex q by doing appropriate single-qubit Clifford Operations at its neighbors.

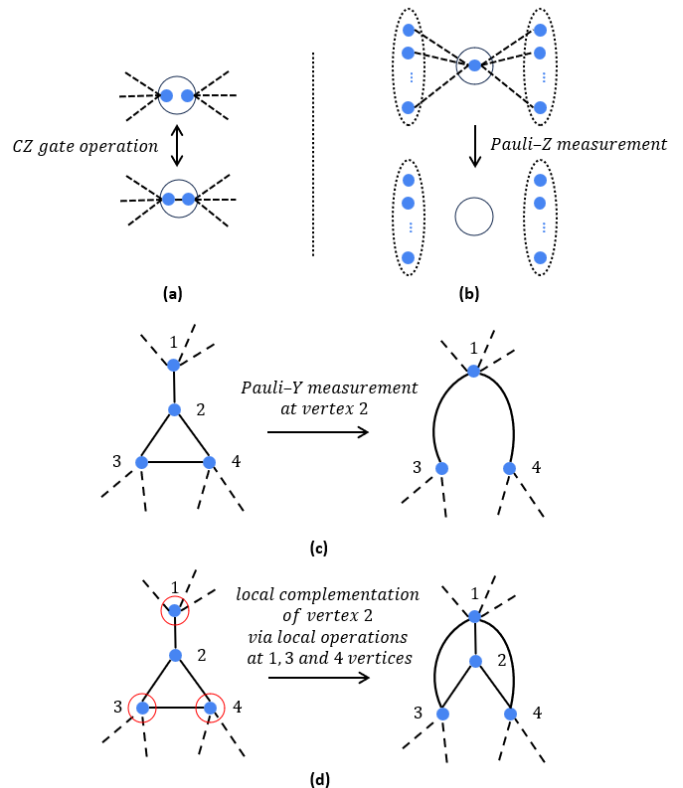


FIGURE 3. Local operations used in our fusion operations.

III. MODEL, PROBLEM, AND RELATED WORKS

In this section, we discuss our network model, formulate the problem addressed, and discuss related work.

Network Model. We denote a quantum network (QN) Q with $V(Q)$ denoting the set of nodes. Adjacent nodes signify nodes connected by a communication link. Our network model is similar to the one used in some of the recent works [12], [28] on efficient generation of EPs and GHZ states. In particular, each node has an atom-photon EP generator with generation latency (t_g) and probability of success (p_g); the atom-photo generation latency refers to the time interval between consecutive attempts by a node to excite an atom for the purpose of generating an atom-photon entangled pair, which implicitly includes other latencies incurred in link EP generation viz. photon transmission, optical-BSM, and classical acknowledgment. A node's atom-photon generation capacity/rate is its aggregate capacity and may be split across its incident links. Each network link $e = (A, B)$ is used

to generate link-EPs, which is locally equivalent to single-edge graph states, using an optical BSM device located in the middle. The optical-BSM has a certain probability of success (p_{ob}); and each half-link (from A or B) to the device has a probability of transmission success (p_e) that decreases exponentially with the link distance. To facilitate atom-atom ES and fusion operations, each network node is also equipped with an atomic-BSM device with appropriate latency and probability of success. There is an independent classical network with a transmission latency of t_c ; we assume classical transmission always succeeds.

Level-Based Fusion Structure.

To maximize the generation rate of a graph state, *multiple concurrent* fusion trees may be required to use all available network resources. Since the number of such trees can be exponential, we use a novel “aggregated” structure that aggregates multiple fusion trees into one structure; we refer to this as a *level-based* structure, as it is composed of multiple levels—with each level consisting of distributed graph states (as vertices) created by fusing states from the previous levels. See Fig. 4. (Similar multi-level structure is used in [29] for generation of EPs.) The bottom level consists of link states, and each non-leaf state S is formed by a fusion of pairs of states in the previous layers; however, there may be several such pairs of states that fuse to create S (in different ways). Each state S may also have multiple “parents” (unlike in a tree), i.e., a state S may be used to create several states in the next layer; in such a case, the generation rate of S is “split” across these fusions. Due to the fusions from previous layers, each vertex/state has a resulting generation rate, estimated as discussed below.

In the level-based structure shown in Fig. 4, node f represents the target graph state we aim to generate, while nodes a , b , and c correspond to single-edge states. Nodes d and e belong to two fusion trees within the same quantum network, both contributing to the generation of f . The two fusion trees are represented by blue and red edges, respectively. The leaf node a has a generation rate of 36 units, which is “split” into 9 and 27 units for the two different fusion operations in the red and blue fusion trees. Each fusion tree individually contributes to generating f with an effective generation rate of 3 units.

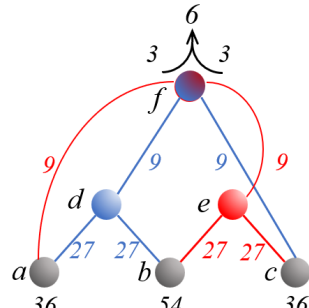


FIGURE 4. Level-based structure. The above structure is an “aggregation” of two fusion trees. The leaf node a ’s generation rate of 36 units is “split” into 9 and 27 for the two different (red and blue) fusion operations. The root node represents the final/target graph state formed in two different ways—for a total generation rate of 6 (3 from each fusion operation). We assume that a parent’s generation rate is $1/3$ of the rate of its children/operands (which are equal).

Graph State Generation Latency/Rate. The expression for estimating the generation rate (or latency) of a state due to a fusion operation in our level-based structure is fundamentally the same as that used in fusion/swapping trees in prior works [12], [28]. Consider a simple case of a non-leaf node t with two children t_l and t_r which are fused to generate t . If the generation events of the children states t_l and t_r are Poisson distributed and thus generation latencies are exponentially distributed, then, the generation latency of the graph state corresponding to t can be estimated as (see [12]):

$$L_t = \left(\frac{3}{2} \max(L_l, L_r) + t_f + t_c\right)/p_f, \quad (1)$$

where L_l and L_r are the generation latencies of the graph states corresponding to the children t_l and t_r , t_f and p_f are the latency and probability of success of the swapping/fusion operation, and t_c is the classical transmission latency which is proportional to the physical distance. The generation rate G_t can thus also be estimated as $G_t = 2/3 \min(G_l, G_r)$, where G_t , G_l , and G_r are the generation rates of the nodes. For a state t generated from multiple pairs of children, we take the sum of the generation rates due to each pair. The generation rate of the leaf vertices (link states) in a level-based structure is given by the generation rate of the EP at the network link. To estimate the generation rate of other states in the structure, we apply the above equation iteratively (for this, we implicitly assume that the resulting latencies also have an exponential distribution).

A. PROBLEM FORMULATION

In this section, we formulate the problem of efficiently generating distributed graph states over a quantum network. Informally, the problem is to generate the level-based structure with a maximum generation output rate, given the constraints of the nodes’ link-EP generation capacity.

Graph State Generation (GSG) Problem. Given a quantum network Q , a graph state G along with its distribution $\tau : V(G) \mapsto V(Q)$, the GSG problem is to determine a level-based structure \mathcal{F} that generates the given distributed graph state with the optimal (highest) generation rate under the following node constraint. (For clarity of presentation, we consider fidelity and decoherence constraints later in §VII.) We refer to the given G as the *target* graph state, and the network nodes $\tau(i)$, for $i \in [1, n]$, as the *terminal* nodes.

Node Constraints. For each network node, the aggregate resources used by \mathcal{F} is less than the available resources. More formally, consider a level-based structure \mathcal{F} . Let \mathcal{E} be the set of all network links, and $\mathcal{E}(i) \subseteq \mathcal{E}$ as the set of links incident on node i . For each link $e \in \mathcal{E}$, let $R(e)$ be the total generation rate of e in \mathcal{F} . Then, the node capacity constraint is formulated as follows.

$$1/t_g \geq \sum_{e \in \mathcal{E}(i)} R(e)/(p_g^2 p_e^2 p_{ob}) \quad \forall i \in V(Q). \quad (2)$$

The above comes from the fact that to generate an edge graph state over e , each end-node of e needs to generate

$1/(p_g^2 p_e^2 p_{ob})$ photons successfully, and that $1/t_g$ is a node's total generation capacity.

GSG Example. Consider the GSG instance shown in Fig. 2. For this instance, one possible solution—a level-based structure not necessarily optimal—is shown in Fig. 5. This structure depicts two ways (shown in blue and red) of generating the distributed graph state x_2-x_4 ; apart from this, the structure is essentially a fusion tree generating the desired (distributed) target graph state from the link states.

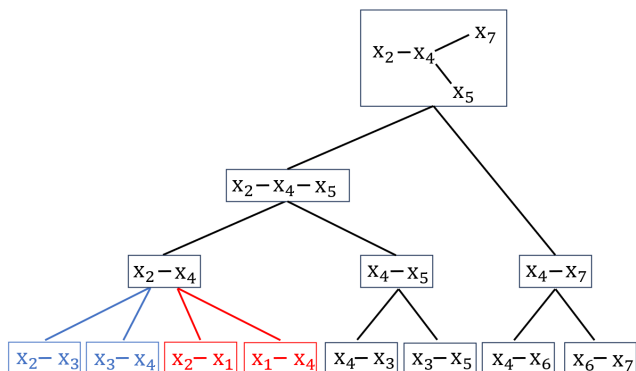


FIGURE 5. A potential solution (not necessarily optimal), a level-based structure, for the network graph and distributed graph state in Fig. 2. The distributed graph state corresponding to a node in the structure is represented by the actual graph and its distribution (e.g., x_2-x_4 represents an edge graph state distributed over nodes x_2 and x_4).

B. RELATED WORKS

There has been recent interest in developing schemes for generating graph states in a quantum network. Most of these works have focused on minimizing the *number* of link EPs consumed in generating the graph states. We discuss these below, categorized by *Centralized* and *Distributed* schemes. To the best of our knowledge, there has been no prior work on efficient generation of arbitrary graph states (or broad classes of graphs) that optimize the generation rates while taking into considering the stochastic nature of the underlying processes; perhaps, the only exception is [28] which considers the generation of GHZ states (we discuss this below, in the last paragraph of this subsection).

Centralized Schemes. In a centralized generation scheme, an appropriately chosen central node first creates the target graph state locally and then teleports qubits to the terminal nodes using EPs. In particular, [23] proposes a max-flow-based approach to minimize the number of link-EPs consumed in generating a graph state using such a scheme. They represent the teleportation routes as multi-path flows and use a network flow approach to maximize the total generation rate. The network-flow approach allows the representation of network resource constraints but ignores the stochastic aspect of the teleportation (or entanglement-swapping) process, which fundamentally requires considering the *length* of the teleportation paths (ignored in the network-flow representation).

Distributed Schemes. In a distributed generation scheme, the target graph state is generated in a distributed manner (perhaps by iteratively merging smaller graph states)—as in the schemes discussed in this paper. In [22], the authors propose a *star expansion* operation/sub-protocol to fuse EPs, and use the operation iteratively to generate a target GHZ (equivalent to a star graph) state. Then, using a succession of such star graphs, they create a complete graph state with appropriate edges “decorated”—which are removed to yield the target graph state. Their optimization objective is the minimization of the number of EPs consumed, and more importantly, for sparse graph states, their scheme can be very wasteful. In a more recent work, [24] presents a graph-theoretic strategy to optimize the fusion-based generation of arbitrary graph states effectively; their strategy comprises three stages: simplifying the graph state, building a fusion tree/network, and determining the order of fusions. They use 3-qubit GHZ states as the basic resource and optimize the number of these states used. They do not discuss techniques to generate and distribute graph states *over a quantum network*; nevertheless, we believe theirs is the most promising approach among existing works for generating arbitrary graph states in a quantum network. Thus, we adapt/extend their scheme for distributing graph states in a quantum network and compare it to our schemes in VIII.

Generating EPs and GHZ States; Our Work. Finally, there have been works on the generation and distribution of specialized graph states, e.g., EPs [12], [25], [26], [29] and GHZ states [28], [30], [31]. Our work on generating general graph states uses a similar network model and optimization formulation as [12], [28], but has different objectives and thus uses different techniques. In particular, [12] designs a dynamic programming approach to construct optimal swapping trees to generate remote EPs, and [28] develops heuristics to construct fusion trees to generate GHZ states. Instead, our objective is to develop a general framework for the optimal generation of general classes of graph states; in particular, we develop a hypergraph-based framework to construct optimal level-based structures (instead of trees) by determining an optimal hyperflow in hypergraphs.

IV. HIGH-LEVEL APPROACH

Here, we discuss our overall approach to optimally solving the GSG problem using linear programming (LP). In the following sections, we will apply our technique to two special cases of graph states: paths and trees. In §VII, we briefly also discuss other classes of graph states to demonstrate the versatility of our approach.

Basic Idea. Given a quantum network and a target graph state, we create a *hypergraph* that has embedded in it all possible level-based structures. In this envisioned hypergraph (see Fig. 6): (i) each vertex is a potential intermediate distributed graph state, (ii) each hyperedge ($\{s_1, s_2\}, s_3$), for vertices s_1, s_2, s_3 , is a fusion operation that fuses graph states s_1 and s_2 to create s_3 , (iii) a “hyperpath” is a potential fusion

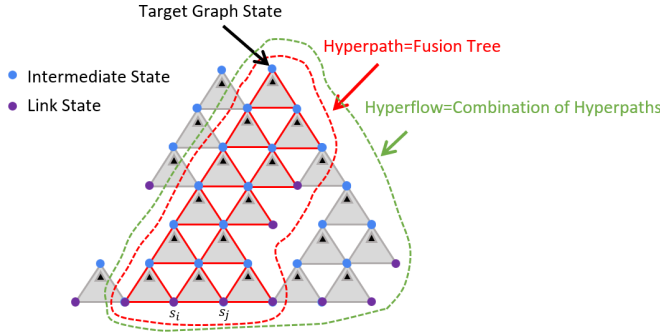


FIGURE 6. A hypergraph over potential intermediate states. Here, each hyperedge represents a fusion operation and a hyperpath represents a potential fusion tree. A hyperflow incorporates multiple hyperpaths, akin to a network flow in a simple graph, and represents a level-based structure.

tree, (iv) and a hyperflow is a level-based structure; here, a hyperflow is basically a “combination” of hyperpaths similar to a network flow in a simple graph being a “combination” of simple paths. To determine the optimal hyperflow (and thus, an optimal level-based structure), we assign flow variables representing generation rates to the hyperedges and create an LP with linear constraints corresponding to network resource constraints, flow conservation, and fusion success probability. This essentially transforms the GSG problem into a max-flow problem on the above hypergraph, where we seek to maximize the flow from the *start* node, which is used to allocate generation rates to single-edge graph states, to the *term* node, which aggregates the generation of all the target graph states. A set of linear equations describes the relationship between the incoming and outgoing flow (generation rate) at each hypervertex, effectively modeling the allowed fusion operations and the corresponding loss in generation rate due to the operation. A similar LP approach has been used as a benchmark in our earlier work [12] for generating EPs.

Key Challenge. In general, *any* distributed graph state in a given network can be considered as an intermediate state in the process of generating a given target graph state; thus, the number of potential intermediate states is exponential ($O(4^n)$) in the number n of network nodes. However, only certain types of intermediate state are likely to be useful/relevant in the generation of a given target state; e.g., to generate a single-edge graph state, it seems reasonable to consider only single-edge graph states as intermediate states (as in the generation of remote EPs via entanglement swapping, which generates only EPs as intermediate states; note that EPs are locally equivalent to single-edge states). Thus, the key challenge in using the above approach is to determine an appropriate set of intermediate states such that the resulting LP over the corresponding hypergraph is computationally feasible and delivers a “good” solution.³ In particular, we also consider the below two-stage approach to

³These good solutions can also be shown to be optimal (as shown in Theorem 1), under appropriate assumptions.

minimize the number of intermediate states considered.

Two-Stage LP Approaches. One strategy we consider to minimize the number of intermediate states considered is to generate the target graph state in two stages: (i) Generate single-edge graph states for each edge in the target graph state G ; (ii) Use these edge graph states to iteratively generate appropriate intermediate states and eventually the target graph state; in this second stage, only the terminal nodes are involved. We discuss such approaches for path and tree graph states in the following sections.

V. GENERATING PATH GRAPH STATES

In this section, we design algorithms to generate distributed path graph states based on the high-level approach described above.⁴ To correspond with the `Two-Stage LP` mentioned later, we refer to this method as `One-Stage LP`. We recall the standard hypergraph notion.

Definition 1: (Hypergraph) A directed hypergraph $H = (V(H), E(H))$ has a set of vertices $V(H)$ and a set of (directed) *hyperedges* $E(H)$, where each hyperedge e is a pair $(t(e), h)$ with the *tail* $t(e) \subset V(H)$ and the *head* $h \in (V(H) - t(e))$.⁵ □

A. OPTIMAL GENERATION OF PATH GRAPH STATES

Consider a GSG problem instance, wherein the target graph state G is a path graph $(1, 2, 3 \dots, n)$ with edges $(i, i + 1)$ for all $1 \leq i < n$ and the target distribution represented by $\tau : V(G) \mapsto V(Q)$.

Basic Idea. For the path state, we hypothesize that the type of intermediate states that can potentially be useful in generating and distributing the path state G are connected subgraphs of G augmented with two edges at the end, i.e., path states $(x, i, i + 1, i + 2, \dots, j, y)$ distributed over $(x, \tau(i), \tau(i + 1), \tau(i + 2), \dots, \tau(j), y)$. (See Theorem 1 for the rationale). We use fusion operations sufficient to build the above states iteratively, starting from the basic link-EP states. This set of intermediate states and fusion operations over them—yields the hypergraph used to develop the linear program for the GSG problem. We start by developing the notation used to define the intermediate states above.

Notation $\langle x, i \cdot j, y \rangle$. Recall that the target graph state G is a path state $(1, 2, 3 \dots, n)$ with the distribution function $\tau()$. We use the notation $\langle x, i \cdot j, y \rangle$, where $1 \leq i \leq j \leq n$ and x, y are vertices in the QN, to represent the path state $(x, i, i + 1, i + 2, \dots, j, y)$ distributed over the network nodes $(x, \tau(i), \tau(i + 1), \tau(i + 2), \dots, \tau(j), y)$. See Fig. 7. The above notation is versatile: i may be equal to j , signifying a path graph state (x, i, y) ; or, the middle parameter $i \cdot j$ may be null (ϕ), signifying an edge graph state (x, y) ; or, x and/or y may be null. To avoid duplicates, we enforce that if $i = j$ or $i \cdot j$ is ϕ , then $\tau'(x) \leq \tau'(y)$ for a distribution mapping τ' .

⁴A path graph state is equivalent to a 1D cluster state in quantum networks

⁵In general hypergraphs, h can also be a subset of $V(H)$, but in our context, h is just a single vertex. Also, in our schemes, $|t(e)|$ is just 1 or 2.

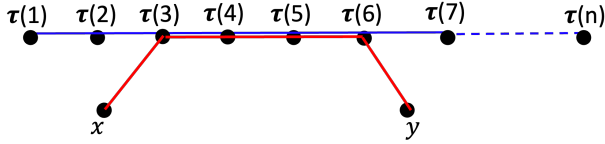


FIGURE 7. Notation $\langle x, 3 \cdot 6, y \rangle$ represents a path state $\langle x, 3, 4, 5, 6, y \rangle$ distributed (shown in red) over the network nodes $\langle x, \tau(3), \tau(4), \tau(5), \tau(6), y \rangle$. The target state $G = (1, 2, \dots, n)$ with distribution function $\tau(\cdot)$ is in blue.

Intermediate States. As mentioned above, for a given target path graph state $(1, 2, \dots, n)$, we choose the following set of (distributed) *intermediate states*: $\langle x, i \cdot j, y \rangle$ with $i, j \in [1, n]$, and x and y being any network nodes. Thus, the total number of intermediate states is approximately $n^2|V(Q)|^2$.

Fusion-Retain and Fusion-Discard Operations. We use fusion operations, viz., *fusion-discard* and *fusion-retain*, to manipulate path graph states. The fusion-discard operation merges path graph states (a_1, a_2, \dots, a_n) and (b_1, b_2, \dots, b_m) to create $(a_1, a_2, \dots, a_{n-1}, b_2, b_3, \dots, b_m)$, if a_n and b_1 are mapped to (i.e., reside at) the same network node. The fusion-retain operation merges path graph states (a_1, a_2, \dots, a_n) and (b_1, b_2, \dots, b_m) to create $(a_1, a_2, \dots, a_n, b_2, b_3, \dots, b_m)$, if a_n and b_1 are mapped to the same network node. See Fig. 8, which also shows the local operations used in the fusion-retain and fusion-discard operations.

Hypergraph. We now construct a hypergraph with the above intermediate states as vertices, with the fusion operations over these vertices yielding the hyperedges, as formally described below. Such a hypergraph embeds all level-based structures that represent a generation of the given target graph state.

Hypergraph Vertices. The hypergraph consists of the following vertices.

- 1) Two distinguished vertices *start* and *term*.
- 2) $Avail(x, i \cdot j, y)$ for each intermediate state $\langle x, i \cdot j, y \rangle$.
- 3) $Prod^r(x, i \cdot j, y)$, and $Prod^d(x, i \cdot j, y)$ vertices for the $Avail(x, i \cdot j, y)$ network nodes, as described below.⁶

Hypergraph Edges. The hyperedges *should* intuitively be of the type $(\{Avail(s_1), Avail(s_2)\}, Avail(s_3))$, signifying the fusion of states s_1 and s_2 to generate s_3 . However, to incorporate the stochasticity of the fusion operations, we create *two* hyperedges: $(\{Avail(s_1), Avail(s_2)\}, Prod^f(s_3))$ and $(Prod^f(s_3), Avail(s_3))$,⁷ where the first edge represents the fusion operation f while the second edge incorporates the fusion's probability of success (see Eqn. 3). See Fig. 9. In

⁶ $Prod(\cdot)$ signifies the graph states *produced* from fusion operations; the generation rate of these states is further reduced, to account for the fusion-success rate, before being made available in the form of $Avail(\cdot)$ for further fusions.

⁷When the hyperedge's head is singleton, we omit the brace brackets. Also, *prod* signifies production, while *Avail* signifies available for consumption.

our context, the superscript f over $Prod(\cdot)$ is d (r) for fusion-discard (fusion-retain). Overall, we have the following set of hyperedges.

- 1) Hyperedges $(start, avail(x, \phi, y))$ for all network links (x, y) , representing generation of link states directly from the network nodes.
- 2) [*fusion_discard*] hyperedges. We create hyperedges for a generation of intermediate states from other states via the fusion-discard operation described above. E.g., by fusing states $\langle x, i \cdot j, z \rangle$ and $\langle z, (j+1) \cdot k, y \rangle$, we get $\langle x, i \cdot k, y \rangle$. Thus, we create the hyperedges:

- $(\{Avail(x, i \cdot j, z), Avail(z, (j+1) \cdot k, y)\}, Prod^d(x, i \cdot k, y))$
- $(Prod^d(x, i \cdot k, y), Avail(x, i \cdot k, y))$

We must also create pairs of hyperedges corresponding to intermediate states with null (ϕ) parameter values. E.g., $\langle x, \phi, z \rangle$ and $\langle z, \phi, y \rangle$ can be fused to get $\langle x, \phi, y \rangle$. We omit stating these cases here for clarity of presentation.

- 3) [*fusion_retain*] hyperedges. Similarly, we create the following hyperedges due to fusion-retain operations. See Fig. 9.

- $(\{Avail(x, i \cdot j, \phi), Avail(\phi, j \cdot k, y)\}, Prod^r(x, i \cdot k, y))$
- $(Prod^r(x, i \cdot k, y), Avail(x, i \cdot k, y))$
- $(\{Avail(x, i \cdot i, \phi), Avail(y, i \cdot i, \phi)\}, Prod^r(x, i \cdot i, y))$
- $(Prod^r(x, i \cdot i, y), Avail(x, i \cdot i, y))$

- 4) Hyperedge $(Avail(\phi, 1 \cdot n, \phi), term)$, signifying generation of the target graph state.

See Fig. 10 for an example hypergraph.

LP Variables. For each hyperedge $e = ((u, v), w)$, we create an LP variable $z_e \in R^+$ which represents the rate of the fusion operation (and thus, its operands and result). This implicitly enforces the (desirable) condition that the generation rates of the states/vertices u and v used for any edge e are equal.

LP Constraints and Optimization Objective.

- **Capacity Constraints:** Each network node x has an atom-photon generation capacity constraint.

$$1/t_g \geq \sum_{(x,y) \in E(Q)} z_{(\{start\}, Avail(x,y))} / (p_g^2 p_e^2 p_{ob}) \quad \forall x \in V(Q)$$

- **Flow Constraints:** Flow constraints vary with vertex types. Let $out(v)$ and $in(v)$ denote the set of outgoing and incoming hyperedges from a vertex v in the hypergraph. Formally, $out(v)$ is $\{e \in E(H) \mid v \in t(e)\}$, and $in(v)$ is $\{e \in E(H) \mid v = h(e)\}$.

– For each vertex v s.t. $v = Avail(\cdot)$:

$$\sum_{e \in in(v)} z_e = \sum_{e' \in out(v)} z_{e'}$$

– For each vertex v s.t. $v = Prod^r(\cdot)$:

$$\sum_{e \in in(v)} \left(\frac{2}{3} p_r\right) z_e = \sum_{e' \in out(v)} z_{e'} \quad (3)$$

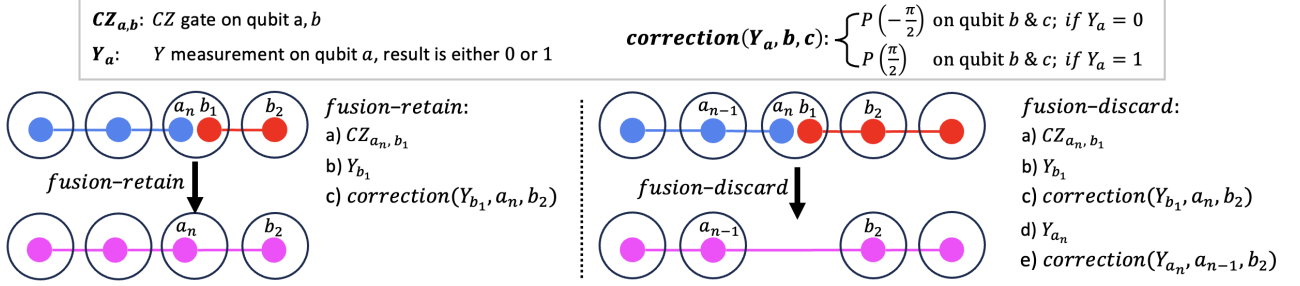


FIGURE 8. Fusion-retain and Fusion-discard operations. They consist of local operations: a CZ gate, Y measurement(s), and phase-shift operations $P(\cdot)$.

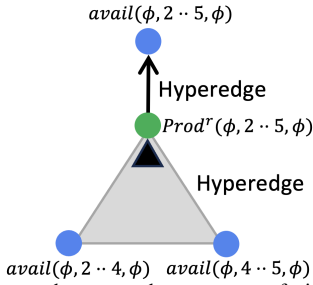
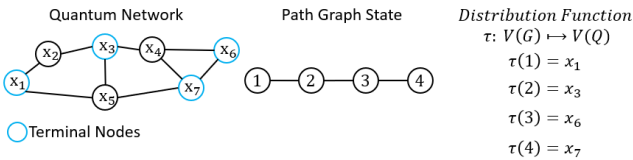
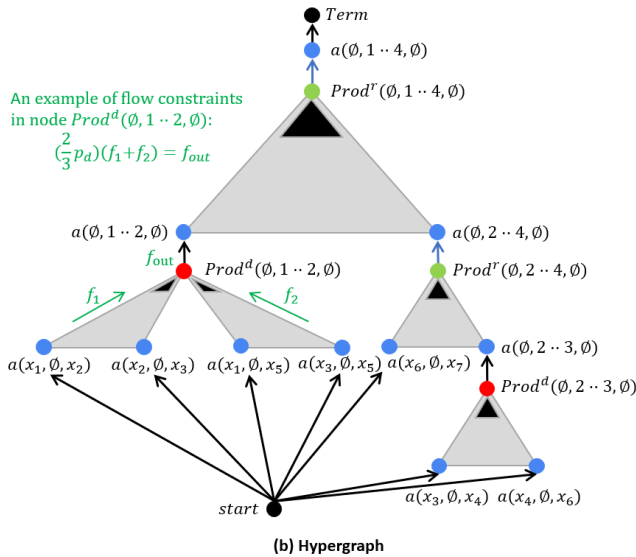


FIGURE 9. Two hyperedges created to represent a fusion-retain operation.



(a) GSG Problem Instance



(b) Hypergraph

FIGURE 10. (a) A GSG problem instance for a path graph state, and (b) A corresponding hypergraph (for sake of clarity, we have shown only a small subset of relevant vertices and hyperedges); here, $a(\cdot) = Avail(\cdot)$.

Here, p_r is the probability of success of the fusion-retain operation.

– For each vertex v s.t. $v = Prod^d(\cdot)$:

$$\sum_{e \in in(v)} \left(\frac{2}{3}p_d\right)z_e = \sum_{e' \in out(v)} z_{e'}$$

Here, p_d is the probability of success of the fusion-discard operation.

• **Objective:** We maximize the sum of the generation rates of the hyperedges incoming into the *term* vertex.

$$\max \sum_{e \in in(term)} z_e$$

Optimality Result. LPs can be solved optimally in polynomial time using interior-point methods or simplex-based approaches. In our evaluations, we use the Gurobi Solver to solve the LPs arising from our problem formulation. We can show the below optimality result.

Theorem 1: The above LP-based algorithm returns an optimal level-based structure for the special-case of the GSG problem wherein the target graph state G is a path graph, and the output level-based structure L is such that: (a) The leaves of L corresponding to all link-EPs; (b) The interior nodes of L corresponding to a (intermediate) distributed graph state H (with a mapping τ) with the following restrictions: (i) The mapping τ is onto, i.e., each qubit in H maps to a unique network node, (ii) H is a connected subgraph (not necessarily induced) of G with additional "leaf" edges (i.e., edges with one of the vertices having a degree of one); (c) The fusion operations used in L are fusion-discard and fusion-retain, with the restriction that the fusion-retain operation is used only at terminal nodes over qubits with degree one in their graph states.

Proof: First, we note that an LP formulation can be optimally solved in polynomial time. Thus, we only need to prove that the hypergraph used to formulate the above-described LP formulation considers all the intermediate graph states and fusion operations among them, under the given restrictions. This, in turn, would imply that any level-based structure L that satisfies the given restrictions is captured by the LP formulation — thus, proving the optimality of the LP-based algorithm.

We can prove by induction (on the number of fusion operations) that any intermediate state formed by the application of the given fusion operations can be represented by our path notation $\langle x, i \cdot \cdot j, y \rangle$ and thus is a vertex in the

LP’s hypergraph. The base case is trivially true, since the notation can represent the link-EPs. Now, let’s consider the given fusion operations applied to two states of the form $\langle x_1, i \cdot j, y_1 \rangle$ and $\langle x_2, r \dots s, y_2 \rangle$ with mappings τ_1 and τ_2 respectively.

(a) **Fusion-Retain Operation.** Let’s first consider the fusion-retain operation. Let $\tau_1(j) = \tau_2(r)$, so that the fusion-retain can be applied to these two qubits at the node $\tau(j)$. This also means that j must be equal to r . As per the given restriction, y_1 and x_2 must be null. Then, the fusion-retain operation leads to the state $\langle x_1, i \dots s, y_2 \rangle$ with an appropriately defined mapping τ_3 . Note that the fusion-retain operation can only be applied to the qubits at terminals and that if $\tau_1(i) = \tau_2(r)$, then application of fusion-retain at $\tau_1(i)$ would need to the violation of the requirement that intermediate states should only have a single qubit at each node.

(b) **Fusion-Discard Operation.** Now, let’s consider the fusion-discard operation. First, we note that the fusion-discard operation can’t occur at the terminal nodes since the fusion-discard at $\tau_1(j) = \tau_2(r)$ (with $j = r$, also) would lead to a state with edge $(j - 1, r + 1) \notin H$; similarly, fusion-discard at a node $\tau_1(k)$, where $i < k < j$ would lead to edges not in H . Now, fusion-discard at $y_1 = x_2$ leads to the state: $\langle x_1, i \dots s, y_2 \rangle$ with an appropriate mapping; note that, in this case, j must be equal to $r - 1$. This holds for all possible cases, viz., when $i = j$ and/or $r = s$, or $x_1 = \phi$ or $y_2 = \phi$.

Finally, it is easy to see that the hyperedges in the hypergraph associated with the LP capture all the fusion operations allowed without violating the given conditions. ■

The theorem and its proof can be generalized to allow for more general fusion operations; in addition, the restriction on fusion-retain operations in the above theorem can also be relaxed but requires tedious analysis. We omit these details for clarity of presentation.

We also note that the performance of the LP-based solution depends greatly on the network topology and other parameters; e.g., if the network topology itself is a path graph, then even a simpler dynamic-programming solution (similar to the one in [12] for EP generation) would be optimal.

B. COMPUTATIONALLY-EFFICIENT LP FORMULATIONS

Even though the above LP formulation is polynomial-time and returns an optimal GSG solution, it can be computationally prohibitive for even moderate-size networks. E.g., for a network of 100 nodes and a path graph state of 10 terminals, the number of intermediate states is about a million, and the LP consists of 100s of millions of variables (from that many hyperedges). Such LP formulations can be computationally infeasible. Thus, we develop the below LP formulations that sacrifice optimality for computational efficiency. In each of the below schemes, the hypergraph is an *induced* sub-hypergraph of the hypergraph from §V-A. Thus, defining the set of intermediate states (and, thus, the hypergraph vertices) for a scheme is sufficient for its full description.

Distance-Based LPs. In this class of LP formulations, we only consider the intermediate states $\langle x, i \cdot j, y \rangle$, where the node $\tau'(x)$ is within a certain distance (physical and/or hop-count)⁸ from the terminal $\tau'(i) = \tau(i)$. The intuition is that intermediate states $\langle x, i \cdot j, y \rangle$ where $\tau'(x)$ is very far away from $\tau(i)$ is unlikely to be helpful in an efficient generation of G . More formally, we impose the condition: $d(\tau'(x), \tau(i)) \leq c \cdot \max(d(\tau(i - 1), \tau(i)), d(\tau(i), \tau(i + 1)))$, where $d()$ is an appropriate distance function and $c > 0.5$.

Left-Sided and Right-Sided LPs. In this scheme, we only consider intermediate states of the type $\langle x, i \cdot j, \phi \rangle$. Similarly, we can consider a scheme that only considers states $\langle \phi, i \cdot j, y \rangle$. We refer to these schemes as Left-Sided LP and Right-Sided LP respectively.

Two-Stage LP. In the Two-Stage LP (see §IV), we generate the target path graph state in two stages. In the first stage, we create the single-edge graph states $\langle \phi, i \cdot (i + 1), \phi \rangle$ for all $i \in [1, n - 1]$ —using the link states and other edge states created in this stage. Then, in the second stage, we create (intermediate) states of the type: $\langle \phi, i \cdot j, \phi \rangle$, eventually yielding the target graph state $\langle \phi, 1 \cdot n, \phi \rangle$ —using only the first-stage edge graph states and second-stage states (and thus, not involving any of the non-terminal nodes). Note that the states considered in the second state are all the connected subgraphs of the target path state, which are $O(n^2)$. See Fig. 11. Another way to look at the above Two-Stage scheme is as follows: Consider the induced subgraph of the hypergraph from §V-A over the vertices of the type $\langle \phi, i \cdot j, \phi \rangle$ or $\langle x, \phi, y \rangle$.

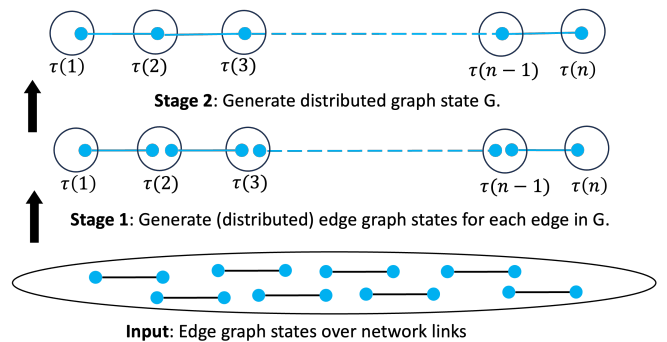


FIGURE 11. Two-Stage Generation Scheme for a Path Graph State $(1, 2, \dots, n)$ with a distribution mapping τ .

Performance Guarantees. We can show the following, similar to the proof of Theorem 1.

Theorem 2: The above Two-Stage scheme returns an optimal solution for the special case of the GSG solution mentioned in Theorem 1 with the additional requirement that the level-based structure L has a “barrier” level (i.e., no state

⁸Technically, we should use a combination of physical distance and hop-count, as both metrics have an impact on the generation rate. In particular, the hop count directly affects the generation gate due to the number of swappings involved, and a longer physical distance can independently entail longer physical distances over individual links in the entanglement path. However, in our evaluations, we only used physical distance since our network instances were spread over a constant area.

at higher level depending on states at lower levels) consisting only of single-edge states corresponding to the edges in G . ■

VI. GENERATING TREE GRAPH STATES

We now design efficient generation schemes to generate tree graph states. Unlike for path graphs, the number of connected induced subgraphs of a tree is exponential in the number of vertices. Thus, considering all connected induced subgraphs (e.g., as in §V-A for paths) is not feasible. In this section, we design two schemes based on a combination of strategies to reduce the number of intermediate states considered.

A. TWO-STAGE GENERATION SCHEME

Consider a GSG problem instance, wherein the target graph state G is a tree T . Recall that the target distribution of G over Q is represented by $\tau : V(G) \mapsto V(Q)$.

Basic Idea. As described in previous sections, a Two-Stage approach consists of two stages. In the first stage, we generate single-edge states corresponding to edges in the target state, and then, in the second stage, we iteratively generate appropriate types of intermediate states and, eventually, the target state. Generally, the natural set of intermediate states to consider in the second stage is the set of all connected subgraphs of the target state (as in §V-B for paths). However, for a tree state, that is exponential. Thus, we consider a carefully chosen set of specialized connected subgraphs such that they are polynomial in number, can be computed from link states via other states from this set (in other words, the set of states yields a connected hypergraph), and is “rich” enough to facilitate an efficient LP solution. We start with a notation that defines these specialized subgraphs of trees.

Notation $\text{Tree}(p, i \cdot j)$. Consider a GSG problem instance: a quantum network (QN) Q and a tree graph state T along with its target distribution $\tau : V(T) \mapsto V(Q)$. Without loss of generality, we *number* the children of each non-leaf node x in T from 1 to $c(x)$, where $c(x)$ is the number of children of x in T . Based on such a numbering, the notation $\text{Tree}(p, i \cdot j)$, where $p \in V(T)$ and $1 \leq i \leq j \leq c(p)$, denotes a distributed tree state T' that is an induced subgraph T' of T containing the following vertices: (i) node p as T' root, (ii) p 's i^{th} to j^{th} children along with *all* their descendants in T . In addition, T' also uses the same distribution function τ over its vertices. See Fig. 12.

Remark. Note that the above tree notation is specifically designed to represent all and only the intermediate states that can arise during the two-stage generation scheme for a given tree graph state. (Later, in §VI-B, we extend/modify our notation to represent intermediate states that can arise in the one-stage generation scheme.) Moreover, the only fusion operations permitted in our schemes are the ones explicitly defined here. Thus, other fusion operations, e.g., fusing one tree’s leaf with another tree’s root, are not allowed.

Overall Two-Stage Scheme. Our Two-Stage generation scheme for the tree graph states consists of two stages.

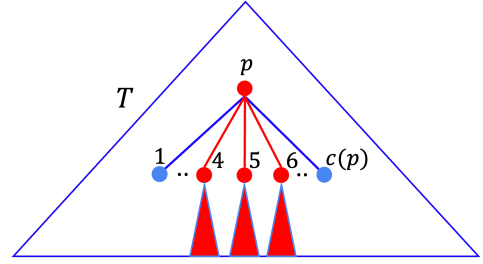


FIGURE 12. Notation $\text{Tree}(p, i \cdot j)$ (here, $i = 4, j = 6$) denotes a distributed graph state T' (shown in red) that is an induced subgraph of T including the interior node p , the i^{th} to j^{th} children of p and all their descendants. The graph state T' has the same distribution function $\tau()$ as T .

- First, from the link states, we generate single-edge graph states corresponding to each edge in T . The intermediate states considered in this stage are single-edge distributed graph states for all pairs of network nodes.
- Then, using the above single-edge states (output of the first stage) and other states generated in this second stage, we iteratively generate intermediate states (which include the target state) of the kind $\text{Tree}(p, i \cdot j)$ where $p \in V(T)$ and $1 \leq i \leq j \leq c(p)$.

For the first stage, we need to use only the *fusion-discard* operation (see §V-A), while for the second stage, we use the following fusion operations (see Fig. 13).

- Fusion_1 : Fuse $\text{Tree}(p, i \cdot j)$ and $\text{Tree}(p, (j + 1) \cdot k)$ to generate $\text{Tree}(p, i \cdot k)$.
- Fusion_2 : Fuse $\text{Tree}(p, 1 \cdot c(p))$ and $\langle p, p' \rangle$ (with p and $p' \in T$ mapped to $\tau(p)$ and $\tau(p')$ respectively) to generate $\text{Tree}(p', i \cdot i)$; here, p is the i^{th} child of p' in T .

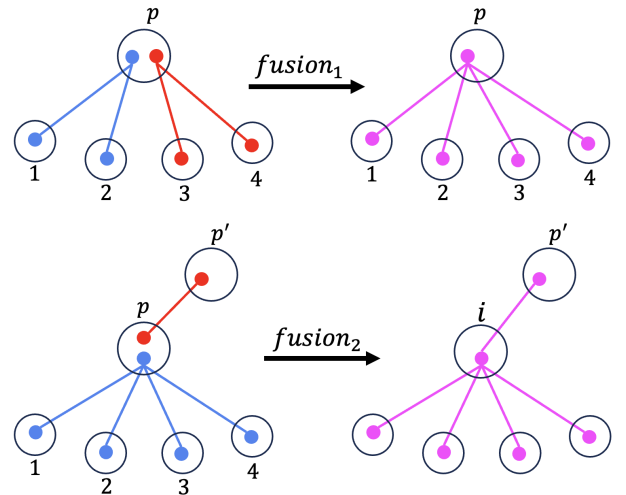


FIGURE 13. Fusion_1 and Fusion_2 operations. Here, the numbers 1 to 4 represent the children numbers of the parent node/vertex p .

Hypergraph and LP. We now construct a hypergraph with $\text{Avail}()$ vertices for all link states and intermediate states (from both stages), and hyperedges to represent the fusion operations over the $\text{Avail}()$ vertices as described above. As

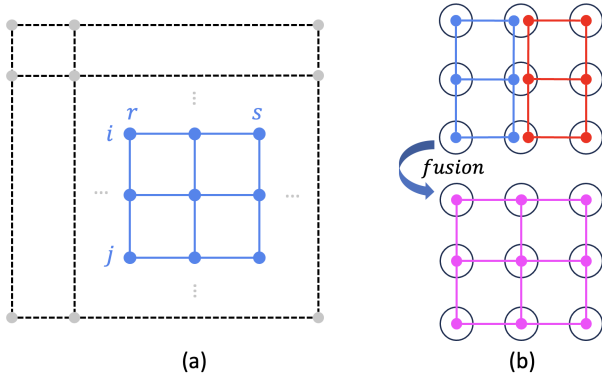


FIGURE 15. Grid graph states. (a) Intermediate state. (b) Fusion-Column Operation.

Grid Graph States. A (m_x, m_y) -grid graph state G with $m_x, m_y \geq 1$ has a 2D structure consisting of m_x columns and m_y rows. For such states, it is natural to consider intermediate states of the type $\langle i \cdot j, r \cdot s \rangle$ consisting of i to j rows and k to l columns of G ; the number of such states is polynomial in the size of G . In addition, we must consider $\langle x, y \rangle$ single-edge graph states where x and y are network nodes. To facilitate the generation of G from these intermediate states, we include fusion-retain, fusion-discard, *fusion-row* and *fusion-column* operations. The fusion-row operation fuses states $\langle i \cdot j, r \cdot s \rangle$ and $\langle j \cdot k, r \cdot s \rangle$ to generate $\langle i \cdot k, r \cdot s \rangle$, and similarly a fusion-column operation fuses states $\langle i \cdot j, r \cdot s \rangle$ and $\langle i \cdot j, s \cdot t \rangle$ to create $\langle i \cdot j, r \cdot t \rangle$. Fig. 15 shows the fusion-column operation. With the above intermediate states and fusion operations, we can construct the hypergraph and formulate an LP as before.

Bipartite Graph States. A (m_a, m_b) -bipartite graph state G has m_a and m_b vertices in the two partitions A and B numbered 1 to m_a and 1 to m_b respectively. To consider a polynomial number of intermediate states, we consider the intermediate states corresponding to the induced subgraphs represented by $\langle i \cdot j, r \cdot s \rangle$ which includes i to j -numbered vertices in A and r to s -numbered vertices in B . See Fig. 16. We include the edge graph states $\langle x, y \rangle$. We can create appropriate fusion operations to generate $\langle i \cdot j, r \cdot s \rangle$ intermediate states; the fusion operations essentially fuse a set of star graphs.

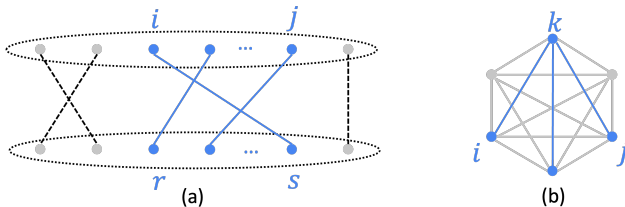


FIGURE 16. Intermediate states of (a) bipartite graph, (b) complete graph.

Complete or Repeater Graph States. For complete graphs G , we can consider the intermediate states as the star graphs $\langle k, i \cdot j \rangle$ with vertex k as its root and vertices numbered i

to j (excluding k) as its children. Along with the single-edge graph states, the total number of intermediate states is polynomial in the size of G , and the only fusion operations needed are fusion-retain, fusion-discard, and a fusion to fuse two star graphs. The above approach easily extends to repeater graphs [24].

Generating Multiple Graph States Concurrently. Our GSG problem considers the generation of (multiple instances of) a single graph state. Our LP formulation can easily be extended to generate several different “types” (including different distributions of the same graph state) of graph states concurrently by essentially creating a hypergraph for each graph state, “merging” the hypergraphs (by taking a union of the vertices and edges, and removing duplicates), and formulating the LP formulation’s objective to maximize the *sum* (or some linear function) of the generation rates of all the graph states.

Decoherence and Fidelity Constraints. To incorporate fidelity and decoherence constraints in the GSG problem formulation, we enforce (as in [12], [28]) that the structure \mathcal{F} satisfy the following constraints: (a) The number of “leaves” of any “tree” in the level-based structure is less than a given threshold τ_l ; this is to limit fidelity degradation due to gate operations. (b) Any qubit’s total memory storage time is less than a given *decoherence threshold* τ_d .

Theoretically, the above constraints on the loss of fidelity due to noisy fusion operations and the age of qubits can be added to the LP formulation as follows, in a way similar to [12] for swapping trees. First, we observe that the fidelity degradation of a generated graph state due to the number of operations can be modeled by limiting the number of its leaf descendants. Second, as observed in [12] for the case of swapping trees, the decoherence constraint (i.e., bounding the total age of a qubit in a graph state) can be incorporated by limiting the *depths* of the left-most and right-most descendants of the children of a graph state in the hypergraph. These *structural* constraints can be enforced in the LP formulation by adding the leaf count and appropriate heights as parameters to $Prod^*$ and $Avail^*$ vertices.

Application to Quantum-Repeater based Quantum Networks. One of the fundamental services that a wide-area QN needs to provide is that of remote or long-distance entanglement generation. In general, entangled graph states have applications in various quantum information processing domains, as discussed in §I. Techniques developed in this work apply to quantum network architectures with quantum repeaters (i.e., nodes with heralded memories and atomic-BSM or entanglement-swapping stations); quantum repeaters are considered essential to enable long-distance or wide-area QNs [32]–[35]. Note that our techniques are independent of how the link-EPs are generated, which may differ across QN architectures [36].

VIII. EVALUATIONS

We now evaluate our schemes and compare them with prior work over the quantum network simulator NetSquid [37].

Graph State Generation Protocol. We build our protocols on top of the link-layer protocol of [38], delegated to continuously generate EPs on a link at a desired rate. The key aspect of our generation protocol is that a fusion operation is done only when *both* the subgraph states (corresponding to the fusion operands) have been generated. On **success** of a fusion operation, the fusion node transmits classical information to the terminal nodes of its sub-states to manipulate the gate operations on their qubits. On fusion **failure**, all the qubits for this graph state will be discarded, allowing the protocols in the lower level to generate new link EPs and subgraphs.

Simulation Setting. We generate random quantum networks in a similar way as in the recent works [12], [25]. By default, we use a network spread over an area of $100km \times 100km$. We use the Waxman model [39], used to create Internet topologies, distribute the nodes, and create links. We select the terminal nodes that store the graph state within the network graph, randomly. The path graph state and tree graph state have the same parameter settings. We vary the number of nodes from 50 to 150 (default being 100) and the number of terminals (i.e., size of the graph state) from 5 to 21 (default being 9). The tree state is as follows: root has 2 children, root's children has 3 children each, and finally, root's grandchildren have 0-3 children each—yielding a tree of size 9 to 21. We vary the edge density from 0.05 to 0.2 with a default value of 0.1. Each data point is for a 100-second duration simulation in NetSquid.

Parameter Values. We use parameter values similar to the ones used in [12], [27]. In particular, we use fusion probability of success (p_f) to be 0.4 and latency (t_f) to be 10μ secs; in some plots, we vary p_f from 0.2 to 0.6. The atomic-BSM probability of success (p_b) and latency (t_b) always equal their fusion counterparts p_f and t_f . The optical-BSM probability of success (p_{ob}) is half of p_b . For generating link-level EPs, we use atom-photon generation times (t_g) and probability of success (p_g) as 50μ sec and 0.33, respectively. Finally, we use photon transmission success probability as $e^{-d/(2L)}$ [27] where L is the channel attenuation length (chosen as 20km for optical fiber) and d is the distance between the nodes. As in [12], [28], we choose a decoherence time of two seconds based on achievable values with single-atom memory platforms [40]; note that decoherence times of even several minutes [41], [42] to hours [43], [44] has been demonstrated for other memory platforms. In NetSquid, the storage noise, channel noise and gate noise are modeled using two parameters: the depolarization rate (for decoherence) and the dephasing rate (for operation-driven) [37]. We choose a depolarization rate of 0.02 and a dephasing rate of 1000 for our experiments.

Prior Algorithms Compared. For comparison with prior work, we implement two schemes for EP and one scheme for GHZ: a recent 3-Star-based scheme from [24] (we

adapt it for a quantum network) and the flow-based approach (called *Central*, here) from [23]. We describe these below. The 3-Star approach [24] is a three-step graph-theoretic scheme: simplify the graph state, decompose the simplified graph into star graphs, and replace each star graph into multiple 3-Star states and determine the order of fusion operations; finally, iterate over the above steps and select the best one. To adapt the scheme to generate graph states in a quantum network, we generate the required 3-Star state locally in a central node, distribute (via teleportation) the qubits of the 3-Star states appropriately, and then fuse them to generate the distributed target graph state. The *Central* approach works by first generating the target graph state locally (at an exhaustively picked optimal central node) and then teleporting the qubits of the graph state to the desired terminals. To continuously generate the graph states at an optimal generation rate, the generation of EPs between C and the terminals is done continuously in parallel with other steps (similar to the max-flow-based approach from [23]).

Our Algorithms. For tree graph states, we implement the One-Stage and Two-Stage schemes. For the path states, the optimal One-Stage scheme (§V-A) takes an exorbitant computation time even for moderate-sized networks; e.g., for a network of 50 (100) nodes, its LP takes 5 (estimated, 120) hours. The Distance-Based approximation schemes perform similarly to One-Stage for $c > 0.8$, but also incur very high computation time. In contrast, the Left-Sided and Right-Sided LP schemes take only a few minutes, even for 100-node networks, and perform close to the optimal One-Stage scheme. See Fig. 19. Based on these observations, we only consider Left-Sided and Two-Stage for path graph states for our further evaluations, which are done over large (default 100 nodes) networks; Right-Sided performs similarly to Left-Sided and, thus, is not shown.

Evaluation Results. We now present the evaluation results comparing prior work with our techniques. Figs.17-18 show the generation rates of various schemes for path and tree graph states, as determined by the NetSquid simulations of at least 100-second duration. In particular, we vary one parameter at a time while keeping the other parameters constant (to their default values). We observe that our schemes outperform the *Central* and 3-Star schemes for both path and tree graph states by up to orders of magnitude; in particular, the performance gap is about **100 (10^6) times** wrt 3-Star (*Central*) for both path and tree graph states of 21 terminals. Among our schemes, as expected, One-Stage outperforms the Two-Stage scheme, sometimes with a smaller margin. Finally, Fig. 20(a)-(b) show the fidelities of the graph states generated in NetSquid simulations by the various schemes for both path and tree graph states for varying fusion success rates. We see that the fidelity of the generated graph states is consistently high. Note that the *Central* approach has high fidelity since we defer generation of the graph state to *after* all the EPs required for teleportation have been successfully generated. We observe that generation rates in

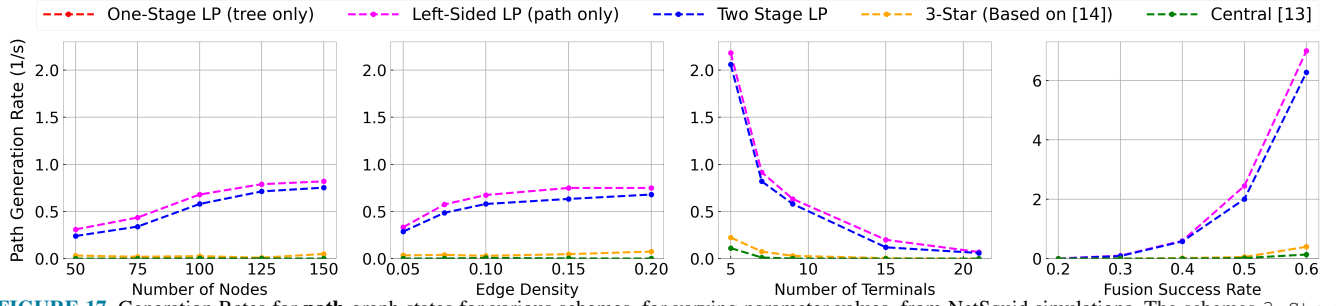


FIGURE 17. Generation Rates for **path** graph states for various schemes, for varying parameter values, from NetSquid simulations. The schemes 3-Star and Central have rates of 0.037-0.0018 and 0.0008-0.002 respectively in (a), rates of 0.075-0.0035 and 0.002-0.001 in (b), rates of 0.224-0.0005 and 0.112-0 in (c), and rates of 0.4-0.0001 and 0.15-0 in (d). (One-Stage takes exorbitant computation time for a 100-node network. Thus, we plot Left-Sided LP scheme; Right-Sided LP scheme has similar performance and not shown.)

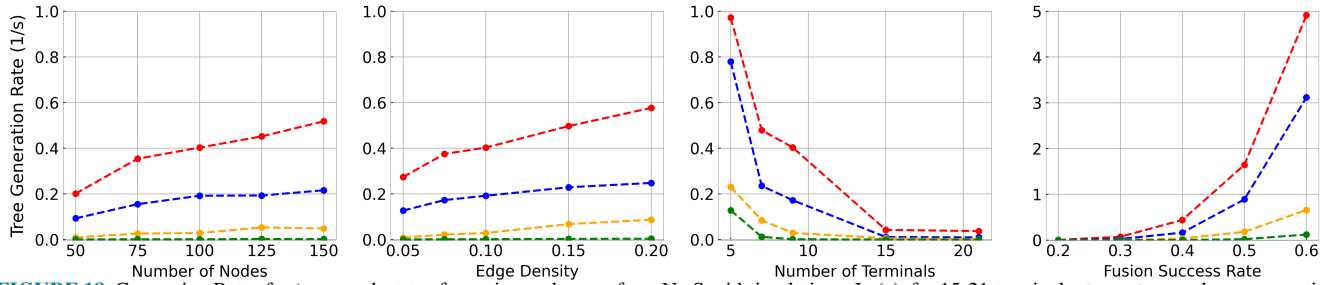


FIGURE 18. Generation Rates for **tree** graph states for various schemes, from NetSquid simulations. In (c), for 15-21 terminals, One-Stage has a generation rate of 0.04-0.03, Two-Stage have rates of 0.009-0.0007, while 3-Star and Central have rates of 0.005-0.0005 and 10^{-6} - 10^{-8} respectively.

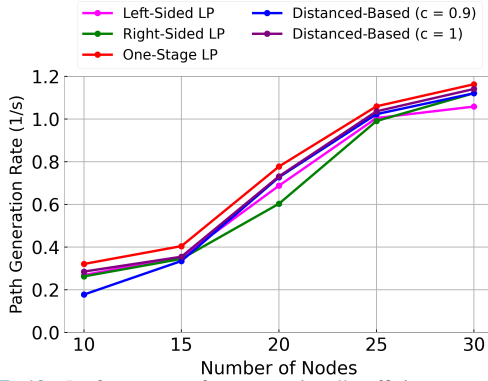


FIGURE 19. Performance of computationally-efficient approximation schemes for path graph states relative to the optimal One-Stage scheme for small network sizes.

the NetSquid simulations show a similar trend as those output by the LP solutions (not shown), but the NetSquid simulation rates were consistently higher; this is because the $2/3$ factor estimation in Eqn. 1 only holds when the operand generation rates are equal—this holds in the LP solution but may not hold at higher levels of the level-based structure in an actual simulation.

Scalability; Runtime. Let n be the number of nodes in the given quantum network, and m be the size (number of terminals) of the target graph state. The complexity of the number of hypervertices and hyperedges in the hypergraph for the LPs used by the various schemes is as follows. (Note that the number of variables in the LP is equal to the number of hyperedges in the corresponding hypergraph). We also

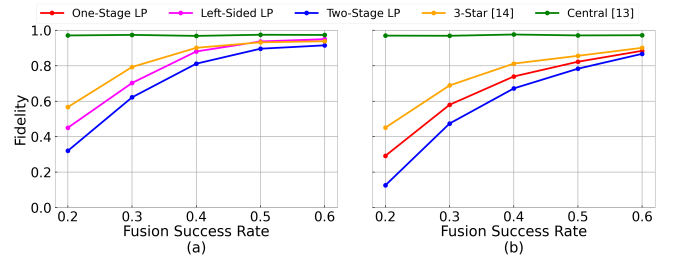


FIGURE 20. Fidelity of generated graph states. (a) Paths, (b) Trees.

give actual numbers for the specific case of the default setting of 100 nodes and 9 terminals.

- One-Stage scheme for paths results in $O(n^2m^2)$ hypervertices and $O(n^3m^4)$ hyperedges. For the default setting, there were 25,808 hypervertices and 814,475 hyperedges.
- Two-Stage scheme for trees results in $O(n^2 + m^3)$ hypervertices and $O(n^3 + m^4)$ hyperedges. For the default setting, there were 10,444 hypervertices and 491,105 hyperedges.
- One-Stage scheme for trees results in $O(n^2 + nm^3)$ hypervertices and $O(n^3 + nm^4 + n^2m^3)$ hyperedges. For the default setting, there were 14,007 hypervertices and 634,970 hyperedges.

In our evaluations, our presented schemes take 1-3 minutes to run for our default settings in 100-node networks on a 5GHz machine (see Table 1). As expected from the above complexity and numbers, our schemes for the generation of tree graph states take less time than those for path graph

states. Overall, the runtime observed is tolerable overhead, especially for the case of continuous generation of graph states as the overhead to determine an efficient generation scheme is only one-time. Note that optimizing generation latency of graph states also minimizes their fidelity degradation due to minimal storage time during generation.

TABLE 1: Average Runtime for Different Schemes in 100-node quantum networks with default settings.)

Scheme	Runtime (seconds)
Tree Graph State Two-Stage	54
Tree Graph State One-Stage	90
Path Graph State Two-Stage	82
Path Graph State Left-Sided One-Stage	126

IX. CONCLUSIONS

We have developed a framework for developing optimized generation and distribution of classes of multipartite graph states under appropriate constraints while considering the stochasticity of the underlying processes. Our methods can also be used to improve the generation rates of given target graph states using desired operations and potential intermediate states (e.g., [45], [46]) by formulating an appropriate LP with the potential intermediate states. Our future work is focused on developing provably optimal generation schemes under fewer assumptions and/or for other useful classes of graph states.

ACKNOWLEDGMENT

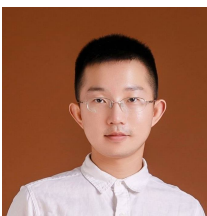
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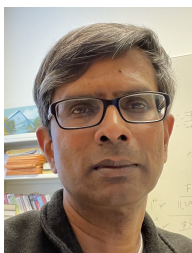


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